

HOMOMESY IN THE GRADED POSET $\Delta(1)$

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ABSTRACT. Among other things, we prove from the scratch that the average value of the size of antichains in any Panyushev orbit of $[m] \times [n]$ is the same and equals $\frac{mn}{m+n}$. This allows us to confirm the second part of Conjecture 5.3 of Panyushev [*Antichains in weight posets associated with gradings of simple Lie algebras*, Math Z 281, pp. 1191–1214, 2015] for \mathfrak{g} being type A , B and C . We raise two conjectures pertaining to the symmetry of Panyushev orbits of $\Delta(1)$ for arbitrary \mathfrak{g} . They indicate that the poset $\Delta(1)$ could be a good place to exhibit the homomesy introduced by Propp and Roby.

1. INTRODUCTION

In 2014, Panyushev formulated several beautiful conjectures concerning the lower ideal generating polynomial, antichain generating polynomial and the reverse operator on certain posets arising from \mathbb{Z} -gradings of Lie algebras [6]. Conjectures 5.1, 5.2, 5.11 and 5.12 there have been considered by Weng and the first named author [2]. The current paper aims to report some progress on Conjecture 5.3 of [6]. Let us warm up with the settings.

Recall that a subset I of a finite poset (P, \leq) is a *lower* (resp., *upper*) *ideal* if $x \leq y$ in P and $y \in I$ (resp. $x \in I$) implies that $x \in I$ (resp. $y \in I$). Let $J(P)$ be the lower ideals of P , partially ordered by inclusion. A subset A of P is an *antichain* if its elements are mutually incomparable. We collect the antichains of P as $\text{An}(P)$. For any $x \in P$, let $I_{\leq x} = \{y \in P \mid y \leq x\}$. Given an antichain A of P , let $I(A) = \bigcup_{a \in A} I_{\leq a}$. The *reverse operator* \mathfrak{X} is defined by $\mathfrak{X}(A) = \min(P \setminus I(A))$. Since antichains of P are in bijection with lower (resp. upper) ideals of P , the reverse operator acts on lower (resp. upper) ideals of P as well.

Now let us introduce the concerned posets $\Delta(1)$. Let \mathfrak{g} be a finite-dimensional simple Lie algebra over the complex numbers. Fix a Cartan subalgebra \mathfrak{h} of \mathfrak{g} . The associated root system is $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}) \subseteq \mathfrak{h}_{\mathbb{R}}^*$. Recall that a decomposition

$$(1) \quad \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$$

is a \mathbb{Z} -grading of \mathfrak{g} if $[\mathfrak{g}(i), \mathfrak{g}(j)] \subseteq \mathfrak{g}(i+j)$ for any $i, j \in \mathbb{Z}$. In particular, in such a case, $\mathfrak{g}(0)$ is a Lie subalgebra of \mathfrak{g} . Since each derivation of \mathfrak{g} is inner, there exists $h_0 \in \mathfrak{g}(0)$ such that $\mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [h_0, x] = ix\}$. The element h_0 is said to be *defining* for the grading (1). Without loss of generality, one may assume that $h_0 \in \mathfrak{h}$. Then $\mathfrak{h} \subseteq \mathfrak{g}(0)$. Let $\Delta(i)$ be the

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set of roots in $\mathfrak{g}(i)$. Then we can choose a set of positive roots $\Delta(0)^+$ for $\Delta(0)$ such that

$$\Delta^+ := \Delta(0)^+ \sqcup \Delta(1) \sqcup \Delta(2) \sqcup \dots$$

is a set of positive roots of $\Delta(\mathfrak{g}, \mathfrak{h})$. Let Π be the corresponding simple roots, and put $\Pi(i) = \Delta(i) \cap \Pi$. Note that the grading (1) is fully determined by $\Pi = \bigsqcup_{i \geq 0} \Pi(i)$. If $|\Pi(1)| = 1$ and $\Pi(i)$ vanishes for $i \geq 2$, we say the \mathbb{Z} -grading (1) is *1-standard*. We refer the reader to Ch. 3, §3 of [4] for generalities on gradings of Lie algebras, see also the paper of Vinberg [12]. Note that each $\Delta(i)$, $i \geq 1$, inherits a poset structure from the usual one of Δ^+ . That is, let α and β be two roots of $\Delta(i)$, then $\alpha \leq \beta$ if and only if $\beta - \alpha$ is a nonnegative integer combination of simple roots. Among the posets $\Delta(i)$, let us focus on $\Delta(1)$ due to its rich structure revealed by the studies [6, 7]. Since now we are working with root posets, after Armstrong, Stump and Thomas [1], we denote the reverse operator \mathfrak{X} by \mathbf{Pan} , and call it the **Panyusev operator**. Indeed, it was Panyushev who firstly observed fascinating properties of this operator on root posets [5]. Note that the lower ideals of $\Delta(1)$ split into disjoint orbits under the action of $\langle \mathbf{Pan} \rangle$, the cyclic group generated by \mathbf{Pan} . We call them **Panyushev orbits**. We denote by $\mathcal{O}(I)$ the Panyushev orbit passing through I .

Now we are able to state Conjecture 5.3 of [6].

Panyushev conjectures. *Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ be any 1-standard \mathbb{Z} -grading of \mathfrak{g} .*

- (a) *Let $d_1 = \max\{\text{ht}(\gamma) \mid \gamma \in \Delta(1)\}$. Then the order of \mathbf{Pan} equals to $d_1 + 1$.*
- (b) *The average value of the size of antichains in any Panyushev orbit is the same and equals $\frac{|\Delta(1)|}{d_1 + 1}$.*
- (c) *The average value of the size of lower ideals in any Panyushev orbit is the same and equals $|\Delta(1)|/2$.*

For convenience, we call part (b) above **Panyushev's antichain conjecture**, and part (c) **Panyushev's ideal conjecture**. In Section 2, we present some preliminary results towards Panyushev conjectures, where the key tool is Lemma 2.1 utilizing Remark 2.3 of [6]. We confirm Panyushev's antichain conjecture for \mathfrak{g} being type *A*, *B* and *C* in Corollary 3.10. Finally, we formulate two conjectures in Section 5: Conjecture 5.1 for lower ideals and Conjecture 5.2 for antichains. They aim to offer a closer look at each Panyushev orbit, thus to offer refinements for Panyushev conjectures. In particular, we will show that Panyushev's ideal conjecture would follow from the conservation law conjecture.

We remark that the cyclic sieving phenomenon (CSP) defined by Reiner, Stanton and White [10] has been established for the triple $(\Delta(1), \mathcal{M}_{\Delta(1)}(t), \langle \mathbf{Pan} \rangle)$ in Theorem 1.7 of [2], based on the results of Rush and Shi [11]. CSP allows one to read the distribution of lengths of Panyushev orbits from the polynomial $\mathcal{M}_{\Delta(1)}(t) := \sum_{I \in J(\Delta(1))} t^{|I|}$. However, it does *not* allow one to read the information, say the size of antichains or lower ideals, *within* an arbitrary Panyushev orbit. Therefore, to handle Panyushev's antichain conjecture for $[m] \times [n]$, we need to develop *new* tools. Here the key is Lemma 3.6, which expresses the antichain sizes for $\mathcal{O}(I)$ in terms of two functions P_I and Q_I associated with I . See (12) and (13) for their precise definitions.

After we finished this work and posted it on arXiv, Thomas kindly brought Propp and Roby's paper [9] to our attention. In particular, Theorem 3.9 had been obtained by them as Theorem 27 of [9]. Comparing with Propp and Roby's proof, our approach builds everything

from the scratch and it allows us to see the Panyushev orbits of $[m] \times [n]$ down to earth. Moreover, it contains ingredients (say the Join-Separate principle in Section 3) that may be developed further. Thus we still include our proof here.

It is interesting to note that Conjectures 5.1 and 5.2 can also be formulated in the language of homomesy due to Propp and Roby [9]. Thus if they hold, the graded poset $\Delta(1)$ would be a good place to exhibit homomesy.

Notation. Let $\mathbb{P} = \{1, 2, \dots\}$. For each $k \in \mathbb{P}$, the poset $[k] := \{1, 2, \dots, k\}$ is equipped with the order-reversing involution c such that $c(i) = k + 1 - i$. For $s, t \in \mathbb{P}$ such that $s < t$, $[s, t] := \{s, s + 1, \dots, t\}$.

2. PRELIMINARY RESULTS

We continue to let \mathfrak{g} be a finite-dimensional simple Lie algebra over \mathbb{C} , and adopt other notation as in Section 1. When the \mathbb{Z} -grading (1) is 1-standard, $\Delta(1)$ has the form

$$(2) \quad [\alpha_i] := \{\alpha \in \Delta^+ \mid [\alpha : \alpha_i] = 1\}.$$

Here α_i is a simple root, and $[\alpha : \alpha_i]$ is the coefficient of α_i in α . An analysis of the type of $\Delta(1)$ can be found in Section 4 of [2], where the minuscule posets classified by Proctor in 1984 [8] play a key role. Let w_0^i be the longest element of the Weyl group of $\mathfrak{g}(0)$ coming from the 1-standard \mathbb{Z} -grading such that $\Pi(1) = \{\alpha_i\}$. Note that $w_0^i(\Delta(1)) = \Delta(1)$. Indeed, w_0^i induces an order-reversing involution on $\Delta(1)$. For any $p \in \Delta(1)$, put $p^* = w_0^i(p)$. Moreover, as in [6], we put $I^* = \Delta(1) \setminus w_0^i(I)$ for any lower ideal I of $\Delta(1)$. Suppose that $\Gamma(I) = \max(I)$ is the antichain corresponding to I , then we put $\Gamma(I)^* = \Gamma(I^*)$.

Lemma 2.1. *Let I be any lower ideal of $\Delta(1)$. Then $\text{Pan}^{-1}(I^*) = \text{Pan}(I)^*$.*

Proof. It suffices to prove this lemma on the antichain level. For any antichain Γ of $\Delta(1)$, similar to Remark 2.3 of [6], $\text{Pan}(\Gamma) = w_0^i(\Gamma^*)$ is a product of two involutions. Indeed,

$$(3) \quad \text{Pan}(\Gamma) = \min(\Delta(1) \setminus I(\Gamma)) = w_0^i(\max(\Delta(1) \setminus w_0^i(I(\Gamma)))) = w_0^i(\Gamma^*).$$

Thus

$$\text{Pan}(\Gamma)^* = (w_0^i(\Gamma^*))^*.$$

On the other hand,

$$\text{Pan}^{-1}(\Gamma^*) = (w_0^i(\Gamma^*))^*.$$

The lemma follows. \square

Remark 2.2. A similar form for the poset $\Delta^+(A_{n-1})$ appeared in Theorem 3.5 of [5].

We say a Panyushev orbit \mathcal{O} is **self-dual** if there exists an ideal I in \mathcal{O} such that I^* belongs to \mathcal{O} as well. Lemma 2.1 guarantees that if \mathcal{O} is self-dual, then for any lower ideal J of $\Delta(1)$, $J \in \mathcal{O}$ if and only if $J^* \in \mathcal{O}$. Since

$$(4) \quad |J| + |J^*| = |\Delta(1)|,$$

we have the following.

Corollary 2.3. *Panyushev's ideal conjecture holds for any self-dual orbit.*

Example 2.4. Let us consider $[2] \times [3]$, which has $\binom{2+3}{2} = 10$ lower ideals. They split into two orbits, each having length 5. One checks easily that all of them are self-dual. \square

Example 2.5. Let us consider $[2] \times [4]$, which has $\binom{2+4}{2} = 15$ lower ideals. They split into three orbits with length 6, 6, 3, respectively. One checks easily that all these orbits are self-dual. \square

Now consider an orbit \mathcal{O} which is not self-dual. Then pick up any lower ideal $I \in \mathcal{O}$, we must have $I^* \notin \mathcal{O}$. We denote the orbit passing through I^* by \mathcal{O}^* , and call it the **dual orbit** of \mathcal{O} .

Corollary 2.6. *Let \mathcal{O} be any Panyushev orbit which is not self-dual, let \mathcal{O}^* be its dual orbit. Then the average value of the size of lower ideals in $\mathcal{O} \cup \mathcal{O}^*$ equals $|\Delta(1)|/2$.*

Proof. As mentioned above, we can pick $I \in \mathcal{O}$ such that $I^* \in \mathcal{O}^*$. Then use Lemma 2.1 and (4). \square

Example 2.7. Let us consider $[3] \times [3]$, which has $\binom{3+3}{3} = 20$ lower ideals. They split into four orbits with length 6, 6, 6 and 2, respectively. One checks easily that the orbit passing through the empty ideal and the orbit with length 2 are self-dual, while the other two orbits are dual to each other. \square

Corollary 2.8. *Let \mathcal{O} be any Panyushev orbit which is not self-dual. Then the average value of the size of antichains in \mathcal{O} equals to that of \mathcal{O}^* .*

Proof. Let Γ be any antichain. Let k be any nonnegative integer. By using (3) $k + 1$ times, we have

$$\text{Pan}^{k+1}(\Gamma) = w_0^i(\text{Pan}^{-k}(\Gamma^*)).$$

Thus

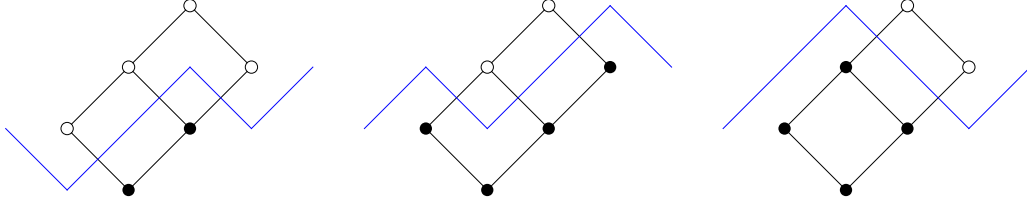
$$|\text{Pan}^{k+1}(\Gamma)| = |\text{Pan}^{-k}(\Gamma^*)|,$$

and the desired result follows. \square

3. RESULTS FOR TYPE A

In this section, we let \mathfrak{g} be A_{m+n} . Then $[\alpha_m]$ is of the form $[m] \times [n]$, and the order-reversing involution w_0^m maps (x, y) to $(m + 1 - x, n + 1 - y)$. We use a Dyck path to separate a lower ideal I from $\Delta(1) \setminus I$. In a Dyck path, we interpret a NE (northeast) step by 1 and a SW (southwest) step by 0. Then a Dyck path becomes a binary word using 0's and 1's. This process associates a binary word $\Theta(I)$ to each $I \in J([m] \times [n])$.

Example 3.1. We present the Dyck paths in blue lines for three lower ideals of $[2] \times [3]$ in Fig. 1. Their associated binary words are 01101, 10110 and 11001, respectively. \square

FIGURE 1. Dyck paths for three ideals of $[2] \times [3]$

Let $B(m, n)$ be the set of binary words with m 1's and n 0's. One sees easily that $\Theta : J([m] \times [n]) \rightarrow B(m, n)$ is a bijection.

For any integer a , we put $1^a := \underbrace{1 \dots 1}_a$, which is interpreted as the empty word if $a \leq 0$. The notation 0^a is defined similarly.

Lemma 3.2. *Let I be a lower ideal of $[m] \times [n]$, and suppose that*

$$(5) \quad \Theta(I) = 1^{a_1} 0^{b_1} \dots 1^{a_i} 0^{b_i} \dots 1^{a_s} 0^{b_s},$$

where $s \geq 2$, each a_i, b_i is positive except for that a_1 and b_s may be 0. Then

$$(6) \quad \Theta(\text{Pan}(I)) = 0^{b_1-1} 1^{a_1+1} \dots 0^{b_i} 1^{a_i} \dots 0^{b_s+1} 1^{a_s-1}.$$

Proof. Without loss of generality, we assume that $s \geq 3$ and focus on a middle part $1^{a_i} 0^{b_i}$, where $1 < i < s$. We draw the corresponding portion of the Dyck path for I in red line in Fig. 2, where dots stand for elements in I , while circles stand for elements in $\Delta(1) \setminus I$. Note that the NE (resp. SW) thick red line segment has length a_i (resp. b_i). Then two elements of the antichain $\Gamma(\text{Pan}(I))$ are shown in blue circles, and the corresponding portion of the Dyck path for $\text{Pan}(I)$ is drawn in blue. Now the SW (resp. NE) thick blue line segment has length b_i (resp. a_i), as desired. We omit the similar analysis for the first part $1^{a_1} 0^{b_1}$ and the last part $1^{a_s} 0^{b_s}$ of $\Theta(I)$. \square

Remark 3.3. Note that in (5), we have $\sum_{i=1}^s a_i = n$ and $\sum_{i=1}^s b_i = m$. Moreover, $s = 1$ in (5) if and only if $\Theta(I) = 1^n 0^m$, if and only if $I = [m] \times [n]$. In this case, $\text{Pan}(I)$ is the empty ideal with the associated binary word $0^m 1^n$.

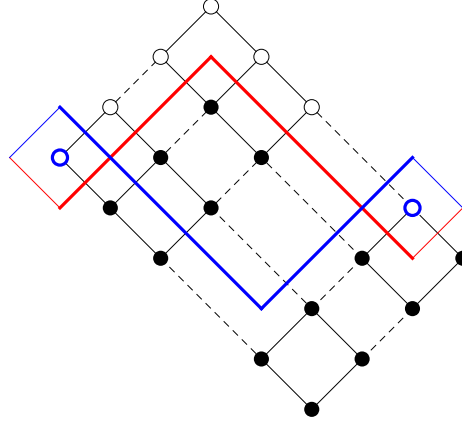


FIGURE 2. Part of the Dyck paths

Let the binary word of I be given by (5). Assume that $a_1 \geq 1$ and $b_s \geq 1$. The pattern of 0's and 1's are

$$(7) \quad - 0^{b_1} - \dots - 0^{b_i} - \dots - 0^{b_s}$$

and

$$(8) \quad 1^{a_1} - \dots - 1^{a_i} - \dots - 1^{a_s} -$$

respectively. There is a unique way to combine (7) and (8) in the zig-zag fashion, after which we recover $\Theta(I)$.

If $b_1 \geq 2$ and $a_s \geq 2$, then by Lemma 3.2, the pattern of 0's becomes

$$(9) \quad 0^{b_1-1} - \dots - 0^{b_i} - \dots - 0^{b_s} \mathbf{0} - .$$

Comparing (9) with (7), one sees that the leftmost 0 has been bumped out, while we add a new 0 from the right. We remember this new 0 as the $(m+1)$ -th, and show it in bold. Note that the $(m+1)$ -th 0 is followed by a $-$. This is not an accident: if one continues the analysis for one more step, one sees that the $(m+2)$ -th 0 will be separated from the $(m+1)$ -th. On the other hand, the pattern of 1's becomes

$$(10) \quad - \mathbf{1} 1^{a_1} - \dots - 1^{a_i} - \dots - 1^{a_s-1}.$$

Comparing (10) with (8), one sees that the rightmost 1 has been bumped out, while we add a new 1 from the left. We remember this new 1 as the $(n+1)$ -th, and show it in bold. Note that the $(n+1)$ -th 1 is preceded by a $-$. Again, this is not an accident: if one continues the analysis for one more step, one sees that the $(n+2)$ -th 1 will be separated from the $(n+1)$ -th 1. Finally, note that there is a unique way to combine (9) and (10) in the zig-zag fashion, after which we recover $\Theta(\text{Pan}(I))$.

Recall that in the above analysis, we have assumed that $a_1 \geq 1$, $b_s \geq 1$, $b_1 \geq 2$ and $a_s \geq 2$. The important thing here is that the assumption “ $a_1 \geq 1$ and $b_s \geq 1$ ” guarantees that $\Theta(I)$ starts with 1 and ends with 0; while the assumption “ $b_1 \geq 2$ and $a_s \geq 2$ ” guarantees that $\Theta(\text{Pan}(I))$ starts with 0 and ends with 1. Of course, there are situations where these

assumptions do not hold. However, in any case, guided by Lemma 3.2, we always have the **Join-Separate Principle**:

- if $\Theta(I)$ ends with 1, separate the $(m+1)$ -th 0 from the m -th 0 by inserting a “–” between them; otherwise, join the $(m+1)$ -th 0 from *right* with the m -th 0 of $\Theta(I)$.
- if $\Theta(I)$ starts with 0, separate the $(n+1)$ -th 1 from the n -th 1 by inserting a “–” between them; otherwise, join the $(n+1)$ -th 1 from *left* with the n -th 1 of $\Theta(I)$.

This principle will be the basic component in the proof of Lemma 3.6.

The following two lemmas read the antichain size $|\Gamma(I)|$ and the ideal size $|I|$ from the binary word $\Theta(I)$.

Lemma 3.4. *Let I be a lower ideal of $[m] \times [n]$ with binary word $\Theta(I)$. Then $|\Gamma(I)|$ equals to the times that “10” occurs in $\Theta(I)$.*

Proof. This follows from considering the Dyck path corresponding to I . \square

Lemma 3.5. *Let I be the lower ideal of $[m] \times [n]$ given by (5). Then $|I| = a_1 b_1 + (a_1 + a_2) b_2 + \dots + (a_1 + \dots + a_s) b_s$.*

Proof. This follows from counting the points under the Dyck path corresponding to I . \square

Suppose that

$$(11) \quad \Theta(I) = 0^{a_1} 1^{b_k - b_{k-1}} 0^{a_2 - a_1} 1^{b_{k-1} - b_{k-2}} \dots 0^{a_k - a_{k-1}} 1^{b_1},$$

where $a_k = m$, $b_k = n$, $a_1 \geq 1$ and $b_1 \geq 1$. Now let us define two functions for I on the interval $[1, m+n]$ which will play an important role in calculating the sizes of antichains in $\mathcal{O}(I)$. Let $A = \{a_1 + 1, \dots, a_k + 1\}$, $B = \{m + 1 + b_1, \dots, m + 1 + b_{k-1}\}$. Put

$$(12) \quad P_I(i) := \begin{cases} 0 & \text{if } i \in A \cup ([m+2, m+n] \setminus B), \\ 1 & \text{if } i \in ([1, m+1] \setminus A) \cup B. \end{cases}$$

Let $C = \{b_1 + 1, \dots, b_k + 1\}$, $D = \{n + 1 + a_1, \dots, n + 1 + a_{k-1}\}$. Put

$$(13) \quad Q_I(i) := \begin{cases} 0 & \text{if } i \in ([1, n+1] \setminus C) \cup D, \\ -1 & \text{if } i \in C \cup ([n+2, n+m] \setminus D). \end{cases}$$

The following lemma will play a key role in handling Panyushev’s antichain conjecture. We postpone its proof to the next section so that one can quickly grasp the sketch.

Lemma 3.6. *Suppose that I is the lower ideal of $[m] \times [n]$ given by (11). Then for $i \in [1, m+n]$, we have*

$$(14) \quad |\Gamma(\text{Pan}^i(I))| = k - 1 + \sum_{j=1}^i (P_I(j) + Q_I(j)).$$

Let us give an example to illustrate the formula (14).

Example 3.7. Fix $m = 3$, $n = 7$. Take $a_1 = 2, a_2 = 3, b_1 = 4, b_2 = 7$. That is, I is the lower ideal of $[3] \times [7]$ with binary word 0011101111. Then one finds that

$$P_I(i) = \begin{cases} 1 & \text{if } i = 1, 2, 8; \\ 0 & \text{otherwise;} \end{cases}$$

and that

$$Q_I(i) = \begin{cases} -1 & \text{if } i = 5, 8, 9; \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, by Lemma 3.2, one calculates the binary words for $\text{Pan}^i(I)$ for $i \in [1, 10]$ as follows:

i	$\Theta(\text{Pan}^i(I))$
1	0101110111
2	1010111011
3	1101011101
4	1110101110
5	1111010011
6	1111100101
7	0111111010
8	1011111100
9	1100011111
10	0011101111

Then by Lemma 3.4 it is direct to check that the formula (14) holds for each $i \in [1, 10]$. \square

Proposition 3.8. Suppose that I is the lower ideal whose binary word $\Theta(I)$ is given by (11). Then

$$\sum_{i=1}^{m+n} |\Gamma(\text{Pan}^i(I))| = mn.$$

Proof. For convenience, we temporarily put $N = m + n$. We have that

$$\begin{aligned} \sum_{i=1}^N |\Gamma(\text{Pan}^i(I))| &= (k-1)N + \sum_{i=1}^N \sum_{j=1}^i (P_I(j) + Q_I(j)) \\ &= (k-1)N + \sum_{i=1}^N (N+1-i)(P_I(i) + Q_I(i)) \\ &= (k-1)N + \sum_{i \in ([1, m+1] \setminus A) \cup B} (N+1-i) - \sum_{i \in C \cup ([n+2, N] \setminus D)} (N+1-i) \\ &= (k-1)N + \sum_{i \in C \cup ([n+2, N] \setminus D)} i - \sum_{i \in ([1, m+1] \setminus A) \cup B} i, \end{aligned}$$

where the first step uses (14), the third step cites (12) and (13), while the last step uses the fact that both $([1, m+1] \setminus A) \cup B$ and $C \cup ([n+2, N] \setminus D)$ have cardinality m . Now the

desired result follows since elementary calculations lead to

$$\sum_{i \in C \cup ([n+2, N] \setminus D)} i - \sum_{i \in ([1, m+1] \setminus A) \cup B} i = mn - (k-1)N.$$

□

Theorem 3.9. (Propp and Roby) *The average value of the size of antichains in any Panyushev orbit of $[m] \times [n]$ equals $\frac{mn}{m+n}$.*

Proof. Take any $I \in J([m] \times [n])$. By Theorem 2 of Fon-Der-Flaass [3], $\mathcal{O}(I)$ has length $(m+n)/d$ for some d dividing both m and n . Thus the multi-set $\{\text{Pan}^i(I) \mid i \in [1, m+n]\}$ is d -copies of the Panyushev orbit $\mathcal{O}(I)$. Hence it is equivalent to verify that

$$(15) \quad \sum_{i=1}^{m+n} |\Gamma(\text{Pan}^i(I))| = mn.$$

When $\Theta(I)$ starts with 0 and ends with 1, say of the form (11), this has been done in Proposition 3.8. Our argument works for the other three cases as well. □

Corollary 3.10. *Panyushev's antichain conjecture holds when \mathfrak{g} is type A, B and C.*

Proof. Let \mathfrak{g} be A_{m+n} , and consider the 1-standard \mathbb{Z} -grading of \mathfrak{g} corresponding to α_m . Then $\Delta(1) = [\alpha_m] \cong [m] \times [n]$, and Theorem 3.9 applies.

Any Panyushev orbit of the poset $H_n := ([n] \times [n])/S_2$ corresponds to an orbit $\mathcal{O}(I)$, where I is a *symmetric* lower ideal of $[n] \times [n]$. That is, $(a, b) \in I$ if and only if $(b, a) \in I$. Thus by Theorem 3.9, Panyushev's antichain conjecture holds for the poset H_n . Due to the structure analysis of $\Delta(1)$ given in Section 4 of [2], we conclude that Panyushev's antichain conjecture holds for \mathfrak{g} being type B and C. □

4. PROOF OF LEMMA 3.6

This section is devoted to proving Lemma 3.6. We adopt the notations in Section 3. We always suppose that (11) holds. Namely, let I be the lower ideal of $[m] \times [n]$ corresponding to the binary word

$$(16) \quad 0^{a_1} 1^{b_k - b_{k-1}} 0^{a_2 - a_1} 1^{b_{k-1} - b_{k-2}} \dots 0^{a_k - a_{k-1}} 1^{b_1},$$

where $a_k = m$, $b_k = n$, $a_1 \geq 1$ and $b_1 \geq 1$.

For any $a \in \mathbb{P}$, we put

$$\underbrace{0-0}_a := \underbrace{0-\dots-0-\dots-0}_a.$$

Note that $\underbrace{0-0}_1 = 0$. The notation $\underbrace{1-1}_a$ is defined similarly. We associate the **long 0-sequence** to I as follows:

$$(17) \quad 0^{a_1} - 0^{a_2 - a_1} - \dots - 0^{a_k - a_{k-1}} - \underbrace{0-0}_{b_1} \underbrace{0-0}_{b_2 - b_1} \dots \underbrace{0-0}_{b_k - b_{k-1}} 0^{a_1} - 0^{a_2 - a_1} - \dots - 0^{a_k - a_{k-1}} -$$

Note that this sequence contains $2m+n$ 0's in total, and ends with $-$. Fix any $i \in [1, m+n]$. Cut out the consecutive segment of (17) starting with the $(i+1)$ -th 0 and ending with the

$(i + m)$ -th 0, and include the “–” left (resp. right) to the $(i + 1)$ -th 0 (resp. $(i + m)$ -th 0) if there is such a “–”. We call this segment the i -th 0– sequence for I .

In a similar fashion, we associate the **long 1– sequence** to I as follows:

$$(18) \quad -1^{b_k-b_{k-1}} - \dots - 1^{b_2-b_1} - 1^{b_1} \underbrace{1-1}_{a_k-a_{k-1}} \dots \underbrace{1-1}_{a_2-a_1} \underbrace{1-1}_{a_1} - 1^{b_k-b_{k-1}} - \dots - 1^{b_2-b_1} - 1^{b_1}$$

Note that this sequence contains $m + 2n$ 1’s in total, and *ends* with –. Here we always read the long 1– sequence and its consecutive segments *from right to left*. For instance, cut out the consecutive segment of (18) starting with the first 1 and the $(n + a_1)$ -th 1, we get

$$\underbrace{1-1}_{a_1} - 1^{b_k-b_{k-1}} - \dots - 1^{b_2-b_1} - 1^{b_1}.$$

Fix any $i \in [1, m + n]$. Cut out the consecutive segment of (18) starting with the $(i + 1)$ -th 1 and ending with the $(i + n)$ -th 1, and include the “–” left (resp. right) to the $(i + n)$ -th 1 (resp. $(i + 1)$ -th 1) if there is such a “–”. We call this segment the i -th 1– sequence for I .

The following lemma reads $\Theta(\text{Pan}^i(I))$ from (17) and (18) for any $i \in [1, m + n]$.

Lemma 4.1. *Let I be the lower ideal of $[m] \times [n]$ given by (16). Fix any $i \in [1, m + n]$. There is a unique way to combine the i -th 0– sequence and the i -th 1– sequence for I in the zig-zag fashion, and one gets $\Theta(\text{Pan}^i(I))$.*

Proof. It amounts to check that (17) and (18) meet the requirements from the scratch. Initially, the first m 0’s in (17) are obtained by replacing each consecutive part of 1’s in (16) with a –. Similarly, the first n 1’s in (18) (counted from right to left) are obtained by replacing each consecutive part of 0’s in (16) with a –. Recall that $a_k = m$ and $b_k = n$.

Suppose that $\Theta(\text{Pan}^{i-1}(I))$ has been settled. Let us consider $\Theta(\text{Pan}^i(I))$. Note that $\Theta(\text{Pan}^{i-1}(I))$ ends with 0 if and only if the $(i - 1)$ -th 1– sequence starts with 1– (recall that we read this sequence from right to left); $\Theta(\text{Pan}^{i-1}(I))$ starts with 1 if and only if the $(i - 1)$ -th 0– sequence starts with –0. Thus the Join-Separate Principle says that

- if the $(i - 1)$ -th 1– sequence starts with “1”, add “–0” to the right side of the $(m + i - 1)$ -th 0; if the $(i - 1)$ -th 1– sequence starts with “1–”, add “0” to the right side of the $(m + i - 1)$ -th 0.
- if the $(i - 1)$ -th 0– sequence starts with “0”, add “1–” to the left side of the $(n + i - 1)$ -th 1; if the $(i - 1)$ -th 0– sequence starts with “–0”, add “1” to the left side of the $(n + i - 1)$ -th 1.

(A). Firstly, let us analyze the pattern of the next n 0’s. Since the first n 1’s (counted from right to left) are of the form

$$-1^{b_k-b_{k-1}} - \dots - 1^{b_2-b_1} - 1^{b_1},$$

the $(m + 1)$ -th 0 to the $(m + n)$ -th 0 are of the form

$$(19) \quad - \underbrace{0-0}_{b_1} \underbrace{0-0}_{b_2-b_1} \dots \underbrace{0-0}_{b_k-b_{k-1}}$$

by the Join-Separate Principle. This agrees with (17) up to the $(m + b_k)$ -th 0.

(B). Secondly, let us switch to analyze the pattern of the next m 1's. Since the first m 0's are of the form

$$0^{a_1} - 0^{a_2-a_1} - \dots - 0^{a_k-a_{k-1}} -$$

the $(n+1)$ -th 1 to the $(n+m)$ -th 1 (counted from right to left) are of the form

$$(20) \quad \underbrace{1-1}_{a_k-a_{k-1}} \dots \underbrace{1-1}_{a_2-a_1} \underbrace{1-1}_{a_1} -$$

by the Join-Separate Principle. This agrees with (18) up to the $(n+a_k)$ -th 1.

(C). Thirdly, let us come back to analyze the pattern of the further next m 0's. Since the $(n+1)$ -th 1 to the $(n+m)$ -th 1 are given by (20), the $(m+n+1)$ -th 0 to the $(m+n+a_k)$ -th 0 are of the form

$$0^{a_1} - 0^{a_2-a_1} - \dots - 0^{a_k-a_{k-1}}$$

by the Join-Separate Principle. This agrees with (17) up to the $(m+b_k+a_k)$ -th 0.

(D). Fourthly, let us switch to analyze the pattern of the further next n 1's. Since the $(m+1)$ -th 0 to the $(m+n)$ -th 0 are of the form (19), the $(n+m+1)$ -th 1 to the $(n+m+n)$ -th 1 (counted from right to left) are of the form

$$1^{b_k-b_{k-1}} - \dots - 1^{b_2-b_1} - 1^{b_1}$$

by the Join-Separate Principle. This agrees with (18) up to the $(n+a_k+b_k)$ -th 1.

(E). Finally, the $(m+n)$ -th 1- sequence starts with 1. Thus we should add “-0” to the $(2m+n)$ -th 0 according to the join-separate principle. This agrees with the last - of (17). Similarly, the $(m+n)$ -th 0- sequence starts with 0. Thus we should add “1-” to the left side of the $(m+2n)$ -th 1 in view of the join-separate principle. This agrees with the last - of (18), and the proof finishes. \square

Remark 4.2. Note that the five steps above are carried out in the zig-zag way.

Example 4.3. Let us revisit Example 3.7, where $a_1 = 2, a_2 = 3, b_1 = 4, b_2 = 7$ and I is the lower ideal of $[3] \times [7]$ with binary word 0011101111. Now the long 0-sequence and 1-sequence for I are

$$00 - 0 - 0 - 0 - 0 - 0 - 00 - 0 - 000 - 0 -$$

and

$$-111 - 111111 - 1 - 111 - 1111,$$

respectively. The third 0-sequence and 1- sequence are

$$-0 - 0 - 0 -, \quad 11 - 1 - 111 - 1.$$

Combining them in the zig-zag fashion gives 1101011101, which agrees with the third row of the table in Example 3.7. Similarly, the tenth 0-sequence and 1- sequence are

$$00 - 0 -, \quad -111 - 1111.$$

Combining them in the zig-zag fashion gives 0011101111, concurring with the last row of the table in Example 3.7. \square

Lemma 4.4. Let I be the lower ideal of $[m] \times [n]$ given by (16). For any $i \in [1, m+n]$, we have

$$(21) \quad |\Gamma(\text{Pan}^i(I))| = |\Gamma(\text{Pan}^{i-1}(I))| + P_I(i) + Q_I(i).$$

Proof. In view of Lemma 3.4 and Lemma 4.1, we should analyze the difference between the number of “10”s in $\Theta(\text{Pan}^i(I))$ and that in $\Theta(\text{Pan}^{i-1}(I))$. Going from $\Theta(\text{Pan}^{i-1}(I))$ to $\Theta(\text{Pan}^i(I))$, we shall *delete* the i -th 1 in (18) and *add* the $(n+i)$ -th 1 in (18) to form the i -th 1- sequence for I .

Deleting the i -th 1 decreases the number of “10”s by one if and only if the i -th 1 has the form 1- in (18). This is measured precisely by the value $Q_I(i)$ defined in (13). In the same way, adding the $(n+i)$ -th 1 increases the number of “10”s by one if and only if the $(n+i)$ -th 1 has the form 1- in (18). This is measured precisely by the value $P_I(i)$ defined in (12). Now (21) follows. \square

Remark 4.5. The proof above tells us that $P_I(i) = -Q_I(i+n)$.

Now (14) follows directly from (21) and Lemma 3.6 is established.

5. TWO CONJECTURES

Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ be any 1-standard \mathbb{Z} -grading. Keep other notations as in Sections 1 and 2. This section aims to formulate two conjectures describing the symmetries *within* each Panyushev orbit in the unifying framework $\Delta(1)$.

For any $p \in \Delta(1)$, and for any Panyushev orbit \mathcal{O} , define $M_{\mathcal{O}}(p)$ to be the number of times that p occurs in the ideals of \mathcal{O} . That is,

$$(22) \quad M_{\mathcal{O}}(p) := |\{I \in \mathcal{O} \mid p \in I\}|.$$

Conjecture 5.1. *Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ be any 1-standard \mathbb{Z} -grading of \mathfrak{g} . Then for any $p \in \Delta(1)$, we have*

$$(23) \quad M_{\mathcal{O}}(p) + M_{\mathcal{O}}(p^*) = |\mathcal{O}|.$$

We note that Panyushev’s ideal conjecture follows from Conjecture 5.1 readily. Indeed, suppose that (23) holds. Then summing both sides over p in $\Delta(1)$ gives

$$\begin{aligned} |\Delta(1)| \cdot |\mathcal{O}| &= \sum_{p \in \Delta(1)} (M_{\mathcal{O}}(p) + M_{\mathcal{O}}(p^*)) \\ &= 2 \sum_{p \in \Delta(1)} M_{\mathcal{O}}(p) \\ &= 2 \sum_{I \in \mathcal{O}} |I|. \end{aligned}$$

Thus

$$\frac{1}{|\mathcal{O}|} \sum_{I \in \mathcal{O}} |I| = \frac{1}{2} |\Delta(1)|,$$

as desired.

For any $p \in \Delta(1)$, and for any Panyushev orbit \mathcal{O} , define $N_{\mathcal{O}}(p)$ to be the number of times that p occurs in the antichains of \mathcal{O} . That is,

$$(24) \quad N_{\mathcal{O}}(p) := |\{\Gamma \in \mathcal{O} \mid p \in \Gamma\}|.$$

Conjecture 5.2. *Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ be any 1-standard \mathbb{Z} -grading of \mathfrak{g} . Then for any $p \in \Delta(1)$, we have*

$$(25) \quad N_{\mathcal{O}}(p) = N_{\mathcal{O}}(p^*).$$

We have verified the two conjectures on many examples.

Finally, we mention that our two conjectures can be formulated in the language of homomesy due to Propp and Roby [9]. Indeed, let 1_p be the function on $J(\Delta(1))$ such that $1_p(I) = 1$ if $p \in I$ and $1_p(I) = 0$ otherwise. Then Conjecture 5.1 amounts to the claim that the function $1_p + 1_{p^*}$ is 1-mesic for any $p \in \Delta(1)$ under the Panyushev operator. Similarly, in the antichain setting, let 1_p be the function on $\text{An}(\Delta(1))$ such that $1_p(A) = 1$ if $p \in A$ and $1_p(A) = 0$ otherwise. Then Conjecture 5.2 is exactly the claim that the function $1_p - 1_{p^*}$ is 0-mesic for any $p \in \Delta(1)$ under the Panyushev operator. Realizing these equivalent formulations has two advantages. Firstly, one sees that Section 4 of Propp and Roby [9] confirms the two conjectures whenever $\Delta(1)$ is a product of two chains. Thus similar to Corollary 3.10, they hold for \mathfrak{g} being types A , B and C . Secondly, verification of the two conjectures in other cases would offer new places exhibiting homomesies.

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